

EXTENSION OF FRAME'S TYPE INEQUALITIES TO BESSEL AND MODIFIED BESSEL FUNCTIONS

KHALED MEHREZ

ABSTRACT. Our aim is to extend some trigonometric inequalities to Bessel functions. Moreover, we extend the hyperbolic analogue of these trigonometric inequalities. As an application of these results we present a generalization of Cusa-type inequality to modified Bessel function. Our main motivation to write this paper is a recent publication of Chen and Sándor, which we wish to complement.

keywords: The Bessel functions, The modified Bessel functions, Frame's type inequalities.

MSC (2010): 33C10; 33C15.

1. Introduction

Bessel and modified Bessel functions of first kinds play an important role in various branches of applied mathematics and engineering sciences. Their properties have been investigated by many scientists and there is a very extensive literature dealing with Bessel functions. In the last few decades many inequalities and monotonicity properties for the functions J_p and I_p and their several combinations have been deduced by many authors, motivated by various problems that arise in wave mechanics, fluid mechanics, electrical engineering, quantum billiards, biophysics, mathematical physics, finite elasticity, probability and statistics, special relativity, etc. For examples, some inequalities for the ratio of $\frac{J_p(x)}{J_p(y)}$ and $\frac{I_p(x)}{I_p(y)}$ was used in 1995 by Sitnik [9]. Recently, Mehrez in [5], by using the monotonicity of the functions $x \mapsto \frac{1-\mathcal{J}_{p+1}(x)}{1-\mathcal{J}_p(x)}$ and $x \mapsto \frac{1-\mathcal{I}_{p+1}(x)}{1-\mathcal{I}_p(x)}$ we give an extension of Huygens types inequalities for Bessel and modified Bessel functions. Moreover, another inequality which is of interest in this paper is discovered in 1954, by Frame [3],

$$(1) \quad \frac{(3 + (x^2/11)) \sinh x}{2 + \cosh x + (x^2/11)} < x < \frac{(3 + (x^2/10)) \sinh x}{2 + \cosh x + (x^2/10)}$$

which holds for for $0 < x < 5$.

Recently, Chen and Sándor [2] extended and sharpened this inequalities as follows:

$$(2) \quad \frac{(3 + \rho_1 x^2) \sinh x}{2 + \cosh x + \rho_1 x^2} < x < \frac{(3 + \rho_2 x^2) \sinh x}{2 + \cosh x + \rho_2 x^2}$$

where $x > 0$ and $\rho_1 = 0$ and $\rho_2 = 1/10$ are the beset possible constants. In addition, this authors we present the trigonometric version of this inequalities as follows, for $0 < x < \pi/2$

$$(3) \quad \frac{(3 - \rho_1 x^2) \sin x}{2 + \cos x + \rho_1 x^2} < x < \frac{(3 - \rho_2 x^2) \sin x}{2 + \cos x - \rho_2 x^2},$$

where $\rho_1 = 1/10$ and $\rho_2 = \frac{8\pi-24}{\pi^3-2\pi^2}$ are the beset possible constants.

In this paper, our aim is to extend the above inequalities to Bessel and modified Bessel functions of the first kinds. For this, let us consider the function $\mathcal{J}_p : \mathbb{R} \rightarrow (-\infty, 1]$, defined by

$$(4) \quad \mathcal{J}_p(x) = 2^p \Gamma(p+1) x^{-p} J_p(x) = \sum_{n \geq 0} \frac{\left(\frac{-1}{4}\right)^n}{(p+1)_n n!} x^{2n}, p > -1,$$

where Γ is the gamma function, $(p+1)_n = \Gamma(p+n+1)/\Gamma(p+1)$ for each $n \geq 0$, is the well-known Pochhammer (or Appell) symbol, and J_p defined by

$$J_p(x) = \sum_{n \geq 0} \frac{(-1)^n (x/2)^{p+2n}}{n! \Gamma(p+n+1)},$$

stands for the Bessel function of the first kind of order p . It is worth mentioning that in particular the function J_p reduces to some elementary functions, like sine and cosine. More precisely, in particular we have:

$$(5) \quad \mathcal{J}_{-1/2}(x) = \sqrt{\pi/2} \cdot x^{1/2} J_{-1/2}(x) = \cos x,$$

$$(6) \quad \mathcal{J}_{1/2}(x) = \sqrt{\pi/2} \cdot x^{-1/2} J_{1/2}(x) = \frac{\sin x}{x},$$

$$(7) \quad \mathcal{J}_{3/2}(x) = 3\sqrt{\pi/2} \cdot x^{-3/2} J_{3/2}(x) = 3 \left(\frac{\sin x}{x^3} - \frac{\cos x}{x^2} \right).$$

For $p > -1$, let us consider $\mathcal{I}_p : \mathbb{R} \rightarrow [1, \infty)$ is defined by

$$\mathcal{I}_p(x) = 2^p \Gamma(p+1) x^{-p} I_p(x) = \sum_{n \geq 0} \frac{\left(\frac{1}{4}\right)^n}{(p+1)_n n!} x^{2n},$$

where I_p is the modified Bessel function of the first kind defined by

$$I_\nu(x) = \sum_{n \geq 0} \frac{(x/2)^{\nu+2n}}{n! \Gamma(\nu+n+1)}, \text{ for all } x \in \mathbb{R}.$$

It is worth mentioning that in particular we have

$$(8) \quad \mathcal{I}_{-1/2}(x) = \sqrt{\pi/2} \cdot x^{1/2} I_{-1/2}(x) = \cosh x,$$

$$(9) \quad \mathcal{I}_{1/2}(x) = \sqrt{\pi/2} \cdot x^{1/2} I_{1/2}(x) = \frac{\sinh x}{x}.$$

The paper is organized as follows: In Section 2 our aim is to extend the inequality (3) to Bessel functions of the first kind. This is motivated by the simple fact that the above inequalities can be rewritten in terms of Bessel functions. In addition, we present also the counterpart of these results for modified Bessel functions of the first kind. In section 3, we would like comment the main results. In particular, we present new Turán type inequality for Bessel function.

In order to study the main results we need the following lemma.

Lemma 1. [8] *Let a_n and b_n ($n = 0, 1, 2, \dots$) be real numbers, and let the power series $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0}^{\infty} b_n x^n$ be convergent for $|x| < R$. If $b_n > 0$ for $n = 0, 1, \dots$, and if $\frac{a_n}{b_n}$ is strictly increasing (or decreasing) for $n = 0, 1, 2, \dots$, then the function $\frac{A(x)}{B(x)}$ is strictly increasing (or decreasing) on $(0, R)$.*

2. Main results

Our first main results is the next Theorem.

Theorem 1. *Let $-1 < p \leq -1/2$ and let $j_{p,1}$ be the first positive zero of the Bessel function J_p . Then the following inequalities holds for all $x \in (0, j_{p,1})$*

$$(10) \quad \frac{\left(\frac{p+2}{p+1} - \alpha x^2\right) \mathcal{J}_{p+1}(x)}{\frac{1}{p+1} + \mathcal{J}_p(x) - \alpha x^2} < 1 < \frac{\left(\frac{p+2}{p+1} - \beta x^2\right) \mathcal{J}_{p+1}(x)}{\frac{1}{p+1} + \mathcal{J}_p(x) - \beta x^2},$$

with best possible constant $\alpha = \frac{1}{8(p+1)(p+3)}$ and $\beta = \frac{\frac{1}{p+1} - \frac{p+2}{p+1}\mathcal{J}_{p+1}(j_{p,1})}{j_{p,1}^2(1 - \mathcal{J}_{p+1}(j_{p,1}))}$.

Proof. Let $-1 < p \leq -1/2$, we consider the function F defined by

$$F(x) = \frac{\frac{1}{p+1} + \mathcal{J}_p(x) - \frac{p+2}{p+1}\mathcal{J}_{p+1}(x)}{x^2(1 - \mathcal{J}_{p+1}(x))} = \frac{f_1(x)}{f_2(x)},$$

where $f_1(x) = \frac{1}{p+1} + \mathcal{J}_p(x) - \frac{p+2}{p+1}\mathcal{J}_{p+1}(x)$ and $f_2(x) = x^2(1 - \mathcal{J}_{p+1}(x))$. By using the differentiation formula [[10], p. 18]

$$\mathcal{J}_p'(x) = -\frac{x}{2(p+1)}\mathcal{J}_{p+1}(x),$$

we get

$$f_1'(x)f_2(x) = \frac{x^3}{2(p+1)}(\mathcal{J}_{p+2}(x) - \mathcal{J}_{p+1}(x) + \mathcal{J}_{p+1}^2(x) - \mathcal{J}_{p+1}(x)\mathcal{J}_{p+2}(x)),$$

and

$$\begin{aligned} f_1(x)f_2'(x) &= \frac{x^3}{2(p+1)}\left(\frac{\mathcal{J}_{p+2}(x)}{p+2} + \frac{(p+1)\mathcal{J}_p(x)\mathcal{J}_{p+2}(x)}{p+2} - \mathcal{J}_{p+1}(x)\mathcal{J}_{p+2}(x)\right) \\ &\quad + 2x\left(\frac{1}{p+1} + \mathcal{J}_p(x) - \frac{p+3}{p+1}\mathcal{J}_{p+1}(x) - \mathcal{J}_p(x)\mathcal{J}_{p+1}(x) + \frac{p+2}{p+1}\mathcal{J}_{p+1}^2(x)\right). \end{aligned}$$

Thus

$$\begin{aligned} H(x) &= x^4(1 - \mathcal{J}_{p+1}(x))^2 F'(x) = \frac{x^3}{2(p+1)}\left(\frac{p+1}{p+2}\mathcal{J}_{p+2}(x) - \mathcal{J}_{p+1}(x) + \mathcal{J}_{p+1}^2(x) - \frac{p+1}{p+2}\mathcal{J}_p(x)\mathcal{J}_{p+2}(x)\right) \\ &\quad + 2x\left(\frac{p+3}{p+1}\mathcal{J}_{p+1}(x) + \mathcal{J}_p(x)\mathcal{J}_{p+1}(x) - \frac{p+2}{p+1}\mathcal{J}_{p+1}^2(x) - \mathcal{J}_p(x) - \frac{1}{p+1}\right) \\ &= A(x) + B(x) + C(x) + D(x), \end{aligned}$$

where $A(x) = \frac{x^3}{2(p+1)}\left(\frac{p+1}{p+2}\mathcal{J}_{p+2}(x) - \mathcal{J}_{p+1}(x)\right)$, $B(x) = \frac{x^3}{2(p+1)}\left(\mathcal{J}_{p+1}^2(x) - \frac{p+1}{p+2}\mathcal{J}_p(x)\mathcal{J}_{p+2}(x)\right)$, $C(x) = 2x\left(\frac{p+3}{p+1}\mathcal{J}_{p+1}(x) - \mathcal{J}_p(x) - \frac{2}{p+1}\right)$ and $D(x) = 2x\left(\mathcal{J}_p(x)\mathcal{J}_{p+1}(x) - \frac{p+2}{p+1}\mathcal{J}_{p+1}^2(x) + \frac{1}{p+1}\right)$. By using power series expansions (4), we find that

$$A(x) = \sum_{n=1}^{\infty} (-1)^n A_n x^{2n+1},$$

where

$$A_n = \frac{n\Gamma(p+1)}{2^{2n-1}\Gamma(n)\Gamma(n+p+2)}, \quad n \geq 1,$$

and

$$C(x) = \sum_{n=1}^{\infty} (-1)^n C_n x^{2n+1},$$

where

$$C_n = \frac{(2-n)\Gamma(p+1)}{2^{2n-1}\Gamma(n+1)\Gamma(n+p+2)}, \quad n \geq 1.$$

Thus

$$\begin{aligned} (11) \quad A(x) + C(x) &= -\frac{\Gamma(p+1)x^3}{\Gamma(p+3)} + \frac{\Gamma(p+1)x^5}{2^2\Gamma(p+4)} - \frac{\Gamma(p+1)x^7}{2^2\Gamma(4)\Gamma(p+5)} + \frac{7\Gamma(p+1)x^9}{2^6\Gamma(5)\Gamma(p+6)} \\ &\quad - \frac{11\Gamma(p+1)x^{11}}{2^8\Gamma(6)\Gamma(p+7)} + \sum_{n=6}^{\infty} (-1)^n \alpha_n x^{2n+1}, \end{aligned}$$

with

$$\alpha_n = \frac{(n^2 - n + 2)\Gamma(p+1)}{2^{2n-1}\Gamma(n+1)\Gamma(n+p+2)}, \quad n \geq 6.$$

Elementary calculations reveal that for $0 < x < j_{p,1}$ and $n \geq 6$,

$$(12) \quad \frac{\alpha_{n+1}(x)}{\alpha_n(x)} = \frac{(n^2 + n + 3)x^2}{4(n+1)(n+p+2)(n^2 - n + 2)} \leq \frac{x^2}{2(n+1)(n+p+2)}.$$

It is also known that for each $p > -1$, we have the following lower and upper bounds for the square of the first positive zero [[6], Eq. 6.8]

$$(13) \quad 4(p+1)\sqrt{p+2} < j_{p,1}^2 < 2(p+1)(p+3).$$

By using the previous inequalities and (13) we have

$$(14) \quad \frac{\alpha_{n+1}(x)}{\alpha_n(x)} \leq \frac{x^2}{2(n+1)(n+p+2)} < \frac{(p+1)(p+3)}{(n+1)(n+p+2)} < 1,$$

for all $n \geq 6$ and $-1 < p < 0$. Therefore, for fixed $x \in (0, j_{p,1})$ and $p \in (-1, 0)$, the sequence $n \mapsto \alpha_n(x)$ is strictly decreasing with regard to $n \geq 6$. It follows from (11)

$$(15) \quad A(x) + C(x) > -\frac{\Gamma(p+1)x^3}{\Gamma(p+3)} + \frac{\Gamma(p+1)x^5}{2^2\Gamma(p+4)} - \frac{\Gamma(p+1)x^7}{2^2\Gamma(4)\Gamma(p+5)} + \frac{7\Gamma(p+1)x^9}{2^6\Gamma(5)\Gamma(p+6)} - \frac{11\Gamma(p+1)x^{11}}{2^8\Gamma(6)\Gamma(p+7)}$$

On the other hand, by using the Cauchy product formula [[10], p. 147],

$$(16) \quad \mathcal{J}_p(x)\mathcal{J}_q(x) = \sum_{n=0}^{\infty} \frac{\Gamma(p+1)\Gamma(q+1)\Gamma(p+q+2n+1)x^{2n}}{2^{2n}\Gamma(n+1)\Gamma(p+q+n+1)\Gamma(p+n+1)\Gamma(q+n+1)}$$

we have

$$(17) \quad \begin{aligned} B(x) &= \frac{x^3}{2(p+1)} \left(\sum_{n=0}^{\infty} \frac{(-1)^n \Gamma^2(p+2)\Gamma(2p+2n+3)x^{2n}}{2^{2n}\Gamma(n+1)\Gamma(2p+n+3)\Gamma(p+n+2)\Gamma(p+n+3)} \right) \\ &= \frac{\Gamma(p+1)x^3}{2\Gamma(p+3)} - \frac{\Gamma(p+1)x^5}{2^2\Gamma(p+4)} + \frac{(2p+5)\Gamma(p+1)x^7}{2^5(p+2)\Gamma(p+5)} - \frac{(2p+7)\Gamma(p+1)x^9}{2^5\Gamma(4)(p+2)\Gamma(p+6)} \\ &\quad + \frac{(2p+7)(2p+9)\Gamma(p+1)x^{11}}{2^7\Gamma(5)(p+2)(p+3)\Gamma(p+7)} - \frac{(2p+9)(2p+11)\Gamma(p+1)x^{13}}{2^8\Gamma(6)(p+2)(p+3)\Gamma(p+8)} + \frac{x^3}{2(p+1)} \sum_{n \geq 6} (-1)^n B_n x^{2n}, \end{aligned}$$

where

$$B_n = \frac{\Gamma^2(p+2)\Gamma(2p+2n+3)}{2^{2n}\Gamma(n+1)\Gamma(2p+n+3)\Gamma(p+n+2)\Gamma(p+n+3)}.$$

By using the inequality (13) and elementary calculations reveal that, for $0 < x < j_{p,1}$ and $n \geq 1$,

$$\begin{aligned}
 \frac{B_{n+1}(x)}{B_n(x)} &= \frac{(2p+2n+3)x^2}{2(n+1)(p+n+3)(2p+n+3)} \\
 &< \frac{(2p+2n+3)j_{p,1}^2}{2(n+1)(p+n+3)(2p+n+3)} \\
 &< \frac{(2p+2n+3)(p+1)(p+3)}{(n+1)(p+n+3)(2p+n+3)} \\
 &< \frac{(2p+2n+3)}{2(2p+n+3)} < 1.
 \end{aligned}
 \tag{18}$$

Therefore, for fixed $x \in (0, j_{p,1})$ and $p \in (-1, 0)$, the sequence $n \mapsto B_n(x)$ is strictly decreasing with regard to $n \geq 1$. It follows from (17) we obtain

$$(19) \quad B_n(x) > \frac{\Gamma(p+1)x^3}{2\Gamma(p+3)} - \frac{\Gamma(p+1)x^5}{2^2\Gamma(p+4)} + \frac{(2p+5)\Gamma(p+1)x^7}{2^5(p+2)\Gamma(p+5)} - \frac{(2p+7)\Gamma(p+1)x^9}{2^5\Gamma(4)(p+2)\Gamma(p+6)}.$$

On the other hand, By using the Cauchy product formula (16) we get

$$\begin{aligned}
 D(x) &= \sum_{n=1}^{\infty} \frac{(-1)^n \Gamma(p+1)\Gamma(p+2)\Gamma(2p+2n+2)(n-2)x^{2n+1}}{2^{2n-1}\Gamma(n+1)\Gamma(2p+n+3)\Gamma(p+n+1)\Gamma(p+n+2)} \\
 (20) \quad &= \frac{\Gamma(p+1)x^3}{2\Gamma(p+3)} - \frac{(2p+7)\Gamma(p+1)x^7}{2^4\Gamma(4)(p+2)\Gamma(p+5)} + \frac{(2p+7)(2p+9)\Gamma(p+1)x^9}{2^5\Gamma(5)(p+3)(p+2)\Gamma(p+6)} \\
 &\quad - \frac{3(2p+9)(2p+11)\Gamma(p+1)x^{11}}{2^7\Gamma(6)(p+3)(p+2)\Gamma(p+7)} + \sum_{n \geq 6} (-1)^n D_n x^{2n+1},
 \end{aligned}$$

where

$$D_n = \frac{\Gamma(p+1)\Gamma(p+2)\Gamma(2p+2n+2)(n-2)}{2^{2n-1}\Gamma(n+1)\Gamma(2p+n+3)\Gamma(p+n+1)\Gamma(p+n+2)}, \quad n \geq 6.$$

From (13) and elementary calculations reveal that, for $0 < x < j_{p,1}$ and $n \geq 6$,

$$\begin{aligned}
 \frac{D_{n+1}(x)}{D_n(x)} &= \frac{(n-1)(2p+2n+3)x^2}{2(n-2)(n+1)(p+n+2)(2p+n+3)} \\
 &< \frac{(n-1)(2p+2n+3)j_{p,1}^2}{2(n-2)(n+1)(p+n+2)(2p+n+3)} \\
 (21) \quad &< \frac{(n-1)(2p+2n+3)(p+1)(p+3)}{(n-2)(n+1)(p+n+2)(2p+n+3)} \\
 &< \frac{(2p+2n+3)(p+3)}{(n-2)(p+n+2)(2p+n+3)} \\
 &< \frac{(2p+2n+3)}{(n-2)(p+n+2)} < \frac{(2p+2n+3)}{4(p+n+2)} < 1.
 \end{aligned}$$

So, for fixed $x \in (0, j_{p,1})$ and $p \in (-1, 0)$, the sequence $n \mapsto D_n(x)$ is strictly decreasing with regard to $n \geq 6$. It follows from (20) we obtain

$$\begin{aligned}
 (22) \quad D(x) &> \frac{\Gamma(p+1)x^3}{2\Gamma(p+3)} - \frac{(2p+7)\Gamma(p+1)x^7}{2^4\Gamma(4)(p+2)\Gamma(p+5)} + \frac{(2p+7)(2p+9)\Gamma(p+1)x^9}{2^5\Gamma(5)(p+3)(p+2)\Gamma(p+6)} \\
 &\quad - \frac{3(2p+9)(2p+11)\Gamma(p+1)x^{11}}{2^7\Gamma(6)(p+3)(p+2)\Gamma(p+7)}.
 \end{aligned}$$

Let $p \in (-1, -\frac{1}{2}]$ and $x \in (0, j_{p,1})$. From (15), (19) and (22), we easily get

$$\begin{aligned}
 (23) \quad H(x) &> -\frac{p(p+5)\Gamma(p+1)x^9}{2^6\Gamma(5)(p+2)(p+3)\Gamma(p+6)} + \frac{\Gamma(p+1)(p^2+5p-6)x^{11}}{2^8\Gamma(5)(p+2)(p+3)\Gamma(p+7)} - \frac{(2p+9)(2p+11)\Gamma(p+1)x^{13}}{2^8\Gamma(6)(p+2)(p+3)\Gamma(p+8)} \\
 &> -\frac{p(p+5)\Gamma(p+1)x^9}{2^6\Gamma(5)(p+2)(p+3)\Gamma(p+6)} + \frac{\Gamma(p+1)x^{11}}{2^8\Gamma(5)(p+2)(p+3)\Gamma(p+7)} \left[p^2 + 5p - 6 - \frac{2(2p+9)(2p+11)(p+1)(p+3)}{5(p+7)} \right] \\
 &> -\frac{p(p+5)\Gamma(p+1)x^9}{2^6\Gamma(5)(p+2)(p+3)\Gamma(p+6)} + \frac{\Gamma(p+1)x^{11}}{2^8\Gamma(5)(p+2)(p+3)\Gamma(p+7)} \left[p^2 + 5p - 6 - \frac{(2p+9)(2p+11)}{12} \right] \\
 &> -\frac{p(p+5)\Gamma(p+1)x^9}{2^6\Gamma(5)(p+2)(p+3)\Gamma(p+6)} + \frac{\Gamma(p+1)(8p^2+20p-171)x^{11}}{2^{10}.3.\Gamma(5)(p+2)(p+3)\Gamma(p+7)} \\
 &> -\frac{p(p+5)\Gamma(p+1)x^9}{2^6\Gamma(5)(p+2)(p+3)\Gamma(p+6)} + \frac{\Gamma(p+1)(8p^2+20p-171)(p+1)x^9}{2^9.3.\Gamma(5)(p+2)\Gamma(p+7)} \\
 &> \frac{\Gamma(p+1)x^9}{2^6\Gamma(5)(p+2)(p+3)\Gamma(p+6)} \left[-p(p+5) + \frac{(8p^2+20p-171)}{96} \right] \\
 &> -\frac{(88p^2+460p+171)\Gamma(p+1)x^9}{2^{11}.3.\Gamma(5)(p+2)(p+3)\Gamma(p+6)} > 0,
 \end{aligned}$$

for all $p \in (-1, -\frac{1}{2}]$ and $x \in (0, j_{p,1})$. Therefore, the function $F(x)$ is increasing on $(0, j_{p,1})$. Furthermore,

$$\lim_{x \rightarrow 0} F(x) = \frac{1}{8(p+1)(p+3)} \text{ and } \lim_{x \rightarrow j_{p,1}} F(x) = \frac{\frac{1}{p+1} - \frac{p+2}{p+1}\mathcal{J}_{p+1}(j_{p,1})}{j_{p,1}^2(1 - \mathcal{J}_{p+1}(j_{p,1}))}.$$

For this the constants $\frac{1}{8(p+1)(p+3)}$ and $\frac{\frac{1}{p+1} - \frac{p+2}{p+1}\mathcal{J}_{p+1}(j_{p,1})}{j_{p,1}^2(1 - \mathcal{J}_{p+1}(j_{p,1}))}$ are the best possible. So, the proof of Theorem 1 is complete. ■

In Theorem 2, we present the version of inequalities (10) for modified Bessel function of the first kind.

Theorem 2. *Let $-1 < p \leq 0$. Then the following inequalities holds for all $x \in (0, \infty)$*

$$(24) \quad \frac{\left(\frac{p+2}{p+1} + \alpha x^2\right) \mathcal{I}_{p+1}(x)}{\frac{1}{p+1} + \mathcal{I}_p(x) + \alpha x^2} < 1 < \frac{\left(\frac{p+2}{p+1} + \beta x^2\right) \mathcal{I}_{p+1}(x)}{\frac{1}{p+1} + \mathcal{I}_p(x) + \beta x^2},$$

with best possible constant $\alpha = 0$ and $\beta = \frac{1}{8(p+1)(p+3)}$.

Proof. Let $-1 < p \leq 0$. We define the function G on $(0, \infty)$ by

$$G(x) = \frac{\frac{1}{p+1} + \mathcal{I}_p(x) - \frac{p+2}{p+1}\mathcal{I}_{p+1}(x)}{x^2(\mathcal{I}_{p+1}(x) - 1)} = \frac{g_1(x)}{g_2(x)},$$

where $g_1(x) = \frac{1}{p+1} + \mathcal{I}_p(x) - \frac{p+2}{p+1}\mathcal{I}_{p+1}(x)$ and $g_2(x) = x^2(\mathcal{I}_{p+1}(x) - 1)$. Thus,

$$g_1(x) = \sum_{n=2}^{\infty} \frac{\Gamma(p+1)(n-1)}{2^{2n}\Gamma(n+1)\Gamma(p+n+2)} x^{2n} = \sum_{n=2}^{\infty} a_n x^{2n},$$

and

$$g_2(x) = \sum_{n=2}^{\infty} \frac{\Gamma(p+2)}{2^{2(n-1)}\Gamma(n)\Gamma(p+n+1)} x^{2n} = \sum_{n=2}^{\infty} b_n x^{2n}.$$

For $-1 < p \leq 0$, the sequence

$$\frac{a_n}{b_n} = \frac{n-1}{4n(p+1)(p+n+1)}, \quad n \geq 2$$

is strictly decreasing. From Lemma 1, the function $G(x)$ is strictly decreasing on $(0, \infty)$. Furthermore,

$$\lim_{x \rightarrow 0} G(x) = \frac{a_2}{b_2} = \frac{1}{8(p+1)(p+3)}$$

and using the asymptotic formula [[1], p. 377]

$$I_\nu(x) = \frac{e^x}{\sqrt{2\pi x}} \left[1 - \frac{4\nu^2 - 1}{1!(8x)} + \frac{(4\nu^2 - 1)(4\nu^2 - 9)}{2!(8x)^2} - \dots \right]$$

which holds for large values of x and for fixed $\nu > -1$, we obtain

$$\lim_{x \rightarrow \infty} G(x) = 0.$$

So, the proof of Theorem 2 is complete. ■

Corollary 1. *Let $-1 < p \leq 0$. Then the following inequalities*

$$(25) \quad \mathcal{I}_{p+1}(x) \leq \frac{1 + (p+1)\mathcal{I}_p(x)}{p+2}$$

$$(26) \quad \frac{\mathcal{I}_{p+1}(x)}{\mathcal{I}_p(x)} + \frac{(p+2)\mathcal{I}_p(x)}{1 + (p+1)\mathcal{I}_p(x)} \geq 2,$$

$$(27) \quad \frac{thx}{x} + \frac{3 \cosh x}{2 + \cosh x} \geq 2,$$

holds for all $x \in \mathbb{R}$.

Proof. Choosing in (24), the values $\alpha = 0$ we conclude that (25) holds. Inequality (26) follows from the arithmetic-geometric mean inequality. By using (8) and (9) in particular for $p = -1/2$ the inequality (26) becomes (27). ■

3. Concluding remarks

In this section we would like to comment the main results of this paper.

1. First note that choosing $p = -1/2$ in (10) and (24), respectively, then we reobtain the inequalities (3) and (2), respectively.

2. Taking $\alpha = \frac{1}{8(p+1)(p+3)}$ and $\beta = \frac{\frac{1}{p+1} - \frac{p+2}{p+1}\mathcal{J}_{p+1}(j_{p,1})}{j_{p,1}^2(1 - \mathcal{J}_{p+1}(j_{p,1}))}$ in (10) we get the inequalities

$$(28) \quad \frac{\frac{1}{p+1} + \mathcal{J}_p(x) - \frac{(\frac{1}{p+1} - \frac{p+2}{p+1}\mathcal{J}_{p+1}(j_{p,1}))x^2}{j_{p,1}^2(1 - \mathcal{J}_{p+1}(j_{p,1}))}}{\frac{p+2}{p+1} - \frac{\frac{1}{p+1} - \frac{p+2}{p+1}\mathcal{J}_{p+1}(j_{p,1})}{j_{p,1}^2(1 - \mathcal{J}_{p+1}(j_{p,1}))}x^2} < \mathcal{J}_{p+1}(x) < \frac{\frac{1}{p+1} + \mathcal{J}_p(x) - \frac{x^2}{8(p+1)(p+3)}}{\frac{p+2}{p+1} - \frac{x^2}{8(p+1)(p+3)}},$$

where $x \in (-j_{p,1}, j_{p,1})$.

3. The inequality (25) is a natural extension of the Cusa type inequality [7].

$$(29) \quad \frac{\sinh x}{x} < \frac{2 + \cosh x}{3}, \quad x > 0.$$

4. In proof of Theorem 1, we can see that the following Turán type inequality

$$(30) \quad \mathcal{J}_{p+1}^2(x) - \frac{p+1}{p+2} \mathcal{J}_p(x) \mathcal{J}_{p+2}(x) > 0$$

holds for all $p \in (-1, 0)$ and $x \in (-j_{p,1}, j_{p,1})$. Indeed, by using the inequality (17), we have

$$(31) \quad \mathcal{J}_{p+1}^2(x) - \frac{p+1}{p+2} \mathcal{J}_p(x) \mathcal{J}_{p+2}(x) = \sum_{n=0}^{\infty} (-1)^n B_n(x)$$

where the sequence $n \mapsto B_n(x)$ is strictly decreasing with regard to $n \geq 1$. In addition, we have

$$(32) \quad \frac{B_1(x)}{B_0(x)} = \frac{x^2}{2(p+3)} < \frac{j_{p,1}^2}{2(p+3)} < p+1 < 1,$$

for all $p \in (-1, 0)$. Thus the sequence $n \mapsto B_n(x)$ is strictly decreasing with regard to $n \geq 0$, that is the Turán-type inequality (30) hold.

5. On the other hand, observe that using (5), (6) and (7) in particular for $p = -1/2$, the Turán type inequality (30) becomes

$$\cos x \left(\frac{\sin x}{x} - \cos x \right) \leq \sin^2 x,$$

which holds for all $x \in (-\pi/2, \pi/2)$.

6. The hyperbolic counterpart of (30) was established in [4] as follows:

$$\mathcal{I}_{p+1}^2(x) - \frac{p+1}{p+2} \mathcal{I}_p(x) \mathcal{I}_{p+2}(x) > 0,$$

where $x \in \mathbb{R}$.

REFERENCES

- [1] M. Abramowitz and I. A. Stegun (eds), Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables (Dover Publications, New York, 1965).
- [2] C.-P. CHen, J. Sándor, sharp inequalities for trigonometric and hyperbolic functions, J. Math. Ineq. V. 9, Nu. 1 (2015), 203217
- [3] J. S. Frame, Some trigonometric, hyperbolic and elliptic approximations, Amer. Math. Monthly 61 (1954), 623626.
- [4] K. Mehrez, Generalized Huygens types inequalities for Bessel and modified Bessel functions of the first kind.
- [5] K. Mehrez, Extension of Huygens type inequalities for Bessel and modified Bessel Functions of the first kinds.
- [6] M.E.H. Ismail and M.E. Muldoon: On the variation with respect to a parameter of zeros of Bessel and q - Bessel functions. J. Math. Anal. Appl. 135(1) (1988), 187-207.
- [7] E. Neuman, J. Sndor, On some inequalities involving trigonometric and hyperbolic functions with emphasis on the Cusa-Huygens, Wilker, and Huygens inequalities. Math. Inequal. Appl. 13(4), 715-723 (2010)
- [8] S. Ponnusamy, M. Vuorinen, Asymptotic expansions and inequalities for hypergeometric functions, Mathematika 44 (1997) 278-301.
- [9] S. M. Sitnik, Inequalities for Bessel functions, Dokl. Akad. Nauk SSSR 340(1) (1995), 2932.
- [10] G. N. Watson, A treatise on the theory of Bessel functions (Cambridge University Press, 1944).

KHALED MEHREZ. DÉPARTEMENT DE MATHÉMATIQUES ISSAT KASSERINE, UNIVERSITÉ DE KAIROUAN, TUNISIA.
E-mail address: k.mehrez@yahoo.fr